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NOTES ON MATRIX THEORY—XIII:

SLIGHTLY INTERTWINED LINEAR PROGRAMMING MATRICES

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SUMMARY

In this paper the functional-equation approach of dynamic programming is used to treat a linear programming problem involving a "slightly intertwined" matrix—i.e., one that is almost block diagonal.

SLIGHTLY INTERTWINED LINEAR PROGRAMMING MATRICES

1. INTRODUCTION

Consider the problem of maximizing the linear form

(1) $$L_N(x) = \sum_{i=1}^{3N} x_i$$

over all x_i satisfying the constraints

(2)
$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \le c_1, \\
&a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \le c_2, \\
&a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + b_1x_4 \le c_3, \\
&a_{44}x_4 + a_{45}x_5 + a_{46}x_6 \le c_4, \\
&a_{54}x_4 + a_{55}x_5 + a_{56}x_6 \le c_5, \\
&a_{64}x_4 + a_{65}x_5 + a_{66}x_6 + b_2x_7 \le c_6, \\
&\vdots \\
&a_{3N-2,3N-2}x_{3N-2} + a_{3N-2,3N-1}x_{3N-1} + a_{3N-2,3N}x_{3N} \le c_{3N-2}, \\
&a_{3N-1,3N-2}x_{3N-2} + a_{3N-1,3N-1}x_{3N-1} + a_{3N-1,3N}x_{3N} \le c_{3N-1}, \\
&a_{3N,3N-2}x_{3N-2} + a_{3N,3N-1}x_{3N-1} + a_{3N,3N}x_{3N} \le c_{3N},
\end{aligned}
$$

and

(3) $$x_i \ge 0.$$

It is assumed throughout that $a_{ij} \ge 0$, $b_i > 0$, $c_i \ge 0$, with a sufficient set of the a_{ij} positive so that the maximum of $L_N(x)$ is not infinite.

We wish to attack this problem—which arises from the study of weakly coupled economic systems, or alternatively from the study of multistage processes with almost-independent stages—by means of the techniques of dynamic programming. Specifically, we shall show that the computational solution can be obtained by means of a sequence of functions of one variable.

It will be clear that a number of similar problems involving almost block-diagonal matrices can be treated by means of the same general method. In another paper, [2], we have illustrated the application of this idea to mechanical and electrical systems in which the matrices are symmetric.

2. DYNAMIC PROGRAMMING FORMULATION

Let us define the sequence of functions of z,

$$(1) \qquad f_0(z) = 0, \; f_K(z) = \max_{x_i} L_K(x), \quad K = 1,2,\ldots,N,$$

where the x_i are subject to the constraints given above, with the exception that the last constraint is now

$$(2) \qquad a_{3K,3K-2}x_{3K-2} + a_{3K,3K-1}x_{3K-1} + a_{3K,3K}x_{3K} \leq z.$$

Employing the principle of optimality, cf. [1], we see that the sequence $\{f_K(z)\}$ satisfies the recurrence relation

$$(3) \quad f_K(z) = \max_{[x_{3K-2},x_{3K-1},x_{3K}]} \Big[x_{3K-2} + x_{3K-1} + x_{3K} + f_{K-1}(c_{3K-3} - b_{K-1}x_{3K-2}) \Big], \; K \geq 1,$$

with the variables x_{3K-2}, x_{3K-1}, x_{3K} subject to the constraints

(4) $$a_{3K-2,3K-2}x_{3K-2} + a_{3K-2,3K-1}x_{3K-1} + a_{3K-2,3K}x_{3K} \leq c_{3K-2},$$

$$a_{3K-1,3K-2}x_{3K-2} + a_{3K-1,3K-1}x_{3K-1} + a_{3K-1,3K}x_{3K} \leq c_{3K-1},$$

$$a_{3K,3K-2}x_{3K-2} + a_{3K,3K-1}x_{3K-1} + a_{3K,3K}x_{3K} \leq z,$$

$$b_{K-1}x_{3K-2} \leq c_{3K-3},$$

$$x_{3K-2}, x_{3K-1}, x_{3K} \geq 0.$$

3. SIMPLIFICATION

We can write the recurrence relation of (2.3) in the form

(1) $$f_K(z) = \underset{x_{3K-2}}{\text{Max}} \left[\underset{x_{3K-1},x_{3K}}{\text{Max}} [\ .\ .\ .\ ] \right]$$

$$= \underset{x_{3K-2}}{\text{Max}} \left[\underset{R_K}{\text{Max}} (x_{3K-1} + x_{3K}) + x_{3K-2} + f_{K-1}(c_{3K-3} - b_{K-1}x_{3K-2}) \right],$$

where R_K is the region in (x_{3K-1}, x_{3K}) space defined by

(2) $$a_{3K-2,3K-1}x_{3K-1} + a_{3K-2,3K}x_{3K} \leq c_{3K-2} - a_{3K-2,3K-2}x_{3K-2},$$

$$a_{3K-1,3K-1}x_{3K-1} + a_{3K-1,3K}x_{3K} \leq c_{3K-1} - a_{3K-1,3K-2}x_{3K-2},$$

$$a_{3K,3K-1}x_{3K-1} + a_{3K,3K}x_{3K} \leq z - a_{3K,3K-2}x_{3K-2},$$

$$x_{3K-1}, x_{3K} \geq 0.$$

Thus we can write

$$(3)\ f_K(z) = \max_{x_{3K-2}} \left[g_K(x_{3K-2}, z) + f_{K-1}(c_{3K-3} - b_{K-1}x_{3K-2}) \right],$$

where x_{3K-2} is constrained by

$$(4)\ 0 \leq x_{3K-2} \leq \text{Min} \left[\frac{c_{3K-3}}{b_{K-1}}, \frac{c_{3K-2}}{a_{3K-2,3K-2}}, \frac{c_{3K-1}}{a_{3K-1,3K-2}}, \frac{z}{a_{3K,3K-2}} \right].$$

The foregoing function $g_K(y,z)$ is readily determined, since the maximum over R_K is attained at a vertex of the region; indeed, for the present problem at most four such vertices need to be considered for any particular assignment of the parameters y, z.

REFERENCES

1. Bellman, R., "The Theory of Dynamic Programming," Bull. Amer. Math. Soc., Vol. 60, 1954, pp. 503-516.

2. ________, Notes on Matrix Theory-XII: Slightly Intertwined Matrices, The RAND Corporation, Paper P-917, 1956.